MOCK EXAM 2025

(1) (i) Let (X, d) be a metric space, and $T : X \to X$ continuous. Define what it means for T to be transitive, or to be minimal.
(ii) Consider $X = S^1 \times S^1$ equipped with the canonical metric induced from $\mathbb{R}^2$, and let $T : X \to X$ be defned by

$$T\big(x, y\big) = \big(x^2, y^3\big).$$

Is T transitive? Is T minimal? Provide detailed arguments for your answers.

(2) Let $T : \mathbb{R} \to \mathbb{R}$ be given by $T(x) = \frac{1}{2}(x - x^{-1})$ if $x \neq 0$ and by $T(0) = 0$.

(i) Show that if μ is the measure given by

$$\mu(A) = \pi^{-1} \int_A (1 + x^2)^{-1}\, dx,$$

then T is measure preserving.
(ii) Show that there is a zero measure set $\mathcal{N} \subset \mathbb{R}_+$ such that for all $x \in \mathbb{R}_+ \backslash \mathcal{N}$, there are infinitely many distinct n_j such that

$$T^{n_j}(x) \in (0, 1).$$

(3) Show that if (X, m) is a probability space and $T : X \to X$ is measurable and measure preserving, then T is ergodic if and only if 1 is a *simple* eigenvalue for the map

$$U_T : L^2(X, m) \to L^2(X, m)$$

given by

$$U_T f = f \circ T.$$

(4) Let $A \in \mathrm{Mat}(n \times n, \mathbb{R})$ a real invertible square matrix, which is hyperbolic, and let

$$\phi(x) = Ax + g(x),\ g \in C^1(\mathbb{R}^n; \mathbb{R}^n)$$

a C^1-diffeomorphism which leaves the origin fixed. State(but do not prove) the Hadamard-Perron theorem. Show that there is a decreasing sequence of open neighborhoods U_j, $j \geq 1$ of the origin, such that

$$\cap_{j=1}^{\infty} U_j = \{0\},$$

and such that the following holds: for $j < k$ and arbitrary $x \in U_j$, either $\phi^l(x) \subset U_j$ for all $l \geq 0$, or else there is some $l_* > 0$ such that

$$\phi^{l_*}(x) \notin U_k.$$

You may refer to the proof of Hadamard-Perron.

(5) (i) Formulate (without proof) the Hartman-Grobman theorem in arbitrary dimension.
(ii) Give an example of a C^1-diffeomorphism $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with hyperbolic fixed point $x_* = 0$ for which there is no C^1-diffeomorphism $h : U \to V$ with U, V open neighborhoods of 0, and such that

$$h^{-1} \circ \phi \circ h = D\phi(0)$$

on U. Provide a detailed argument.

(6) (i) State and prove the Poincaré recurrence theorem.
(ii) Let (X, d) be a metric space which comes equipped with a finite measure m, such that all open sets U are measurable. Assume that (X, d) admits a countable base. Show that for almost all $x \in X$, we have

$$\liminf_{n\to\infty} d(x, T^n x) = 0.$$

(7) (i) Formulate and prove the von Neumann mean ergodic theorem.
(ii) Let (X, m) be a finite measure space, and let $T : X \to X$ be measurable and measure preserving. Show that if $f \in L^3(X, m)$, then the limit

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) =: f_*(x)$$

exists in the L^3-sense.

(8) Let $A = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$. Show that the map $x \to Ax$ from $\mathbb{R}^2$ to itself induces a map

$$T : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2.$$

Is this map transitive? Determine the global stable and unstable manifolds $W^\pm$ of $0 \in \mathbb{R}^2/\mathbb{Z}^2$, where

$$W^\pm(0) = \{p \in \mathbb{R}^2/\mathbb{Z}^2,\, T^{\pm n} p \longrightarrow 0\},$$